## Sign Change Methods

**Example**

Show that $f(x) = x^3 - 3x + 1$ has a root between $x = 1$ and $x = 2$.

**Example**

How can a sign change method fail?

**Tip**

When you have found an approximate root, you must *verify* it to the required accuracy using a sign change. For example, to show that $\alpha = 1.35$ to 2 decimal places, check that $f(1.345)$ and $f(1.355)$ have opposite signs.

**Fact** (Intermediate Value Theorem) — If $f$ is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then $f$ has at least one root in $(a, b)$.

# Bisection Method

**Fact** (Bisection) — Given an interval $[a,b]$ with $f(a)$ and $f(b)$ of opposite sign:

1. Find the midpoint $m = \frac{a+b}{2}$.
2. Evaluate $f(m)$.
3. If $f(a)$ and $f(m)$ have opposite signs, the root is in $[a,m]$. Otherwise it is in $[m,b]$.
4. Repeat with the new interval.

Each step halves the width of the interval, so after $n$ steps the root is known to within $\frac{b-a}{2^n}$.

**Example**

Use the bisection method to find the root of $f(x) = x^3 - 3x + 1$ in $[1,2]$, correct to 1 decimal place.

## Decimal Search

**Fact** (Decimal Search) — Starting from an interval $[a, b]$ containing a root:

1. Divide $[a, b]$ into 10 equal sub-intervals.
2. Evaluate $f$ at each division point to find the sub-interval where the sign changes.
3. Repeat with the narrower interval.

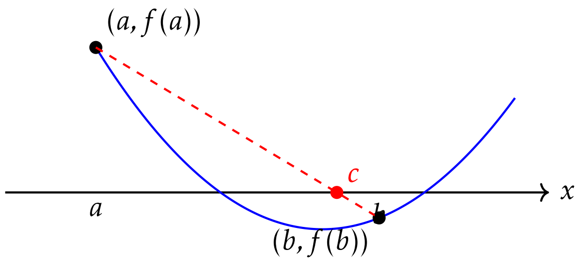
Each stage gives one more decimal place of accuracy.

**Example**

Use decimal search to find the root of $f(x) = x^3 - 3x + 1$ in $[1, 2]$ correct to 2 decimal places.

## Linear Interpolation

Instead of taking the midpoint (bisection), we can draw the straight line between $(a, f(a))$ and $(b, f(b))$ and use its $x$-intercept as our next approximation:




> **Example**
> Derive a formula for $c$, the $x$-intercept of the line through $(a, f(a))$ and $(b, f(b))$.

**Example**

Use linear interpolation to find a better approximation to the root of $f(x) = x^3 - 3x + 1$ in $[1, 2]$.

**Tip**

Linear interpolation usually converges faster than bisection, but it is not guaranteed to. Bisection is slower but *always* halves the interval.

## Iterative Methods

**Fact** (Fixed-Point Iteration) — To solve $f(x) = 0$, rearrange to the form $x = g(x)$. Then the iteration

$$x_{n+1} = g(x_n)$$

may converge to a root, provided $|g'(\alpha)| < 1$ near the root $\alpha$.

**Example**

We want to solve $x^3 - 3x + 1 = 0$ near $x = 1.5$. Consider two rearrangements:

(a) $x = \frac{x^3 + 1}{3}$

(b) $x = \sqrt[3]{3x - 1}$

Starting from $x_0 = 1.5$, find the first 5 iterates of each. Which converges?

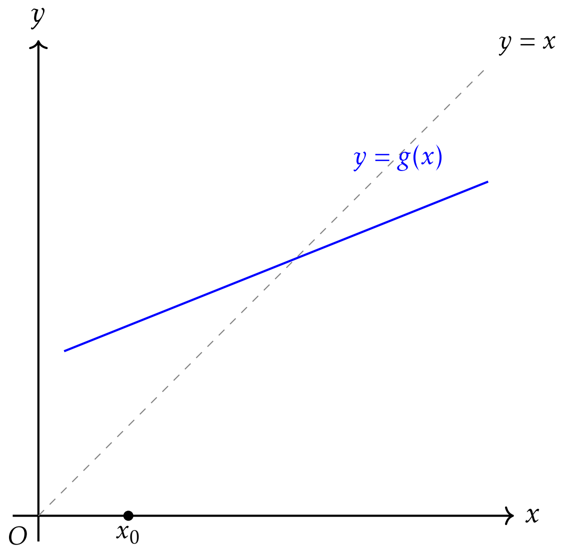
**Tip**

The same equation can be rearranged in many ways — some converge and some don't. If one rearrangement diverges, try another!

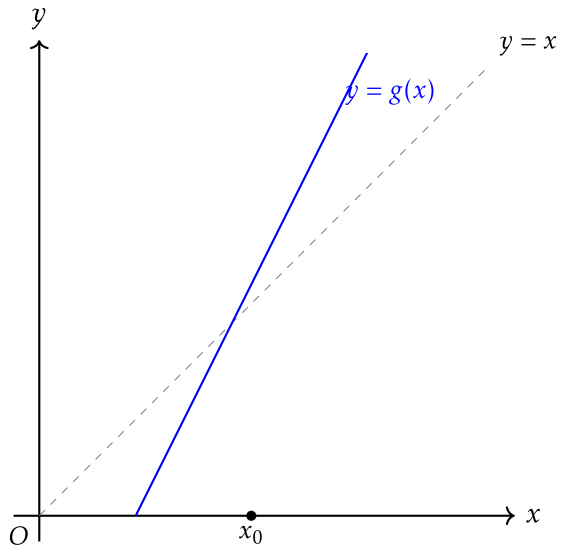
**Example**

The iteration $x_{n+1} = \sqrt[3]{3x_n - 1}$ appears to converge to 1.5320… Show that the root is 1.532 correct to 3 decimal places.

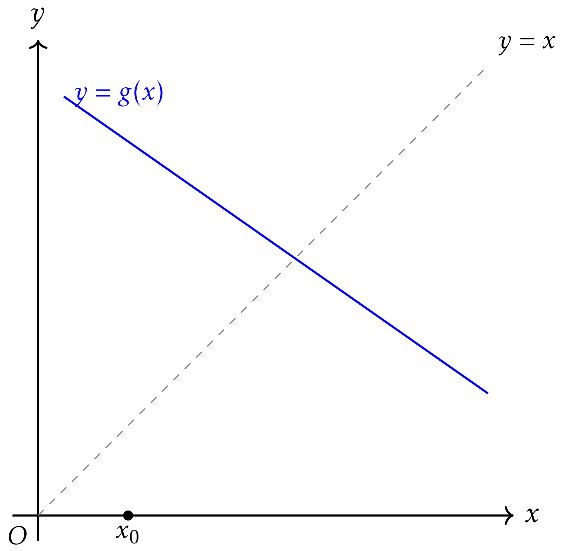
## Staircase and Cobweb Diagrams



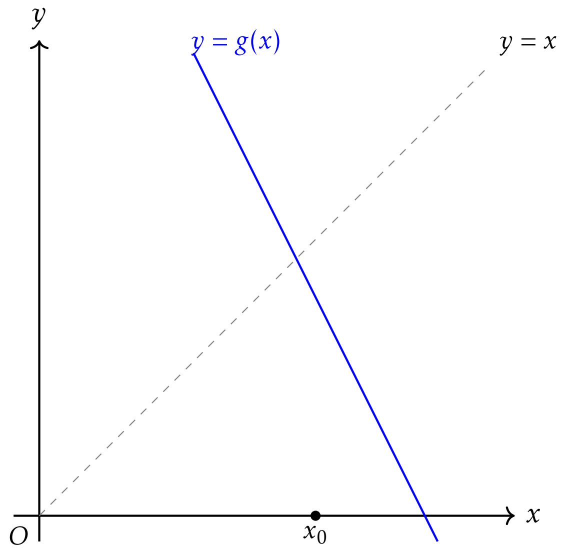

**(a)**




**(b)**




**(c)**




**(d)**

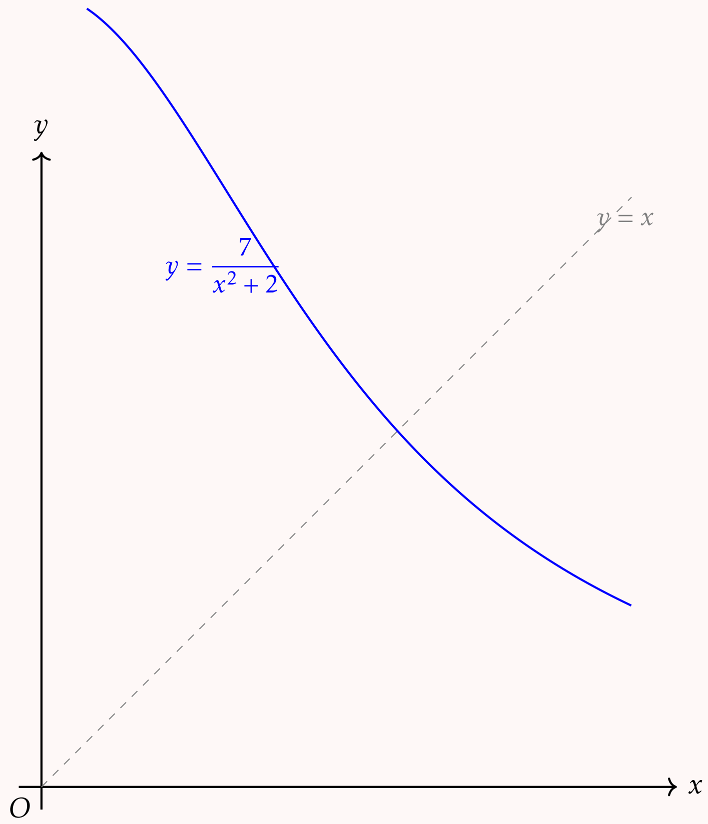
**Example** (OCR FP1 June 2014 Q4, adapted)

The equation $x^3 + 2x - 7 = 0$ has a root $\alpha$ in the interval $[1, 2]$.

(i) Show that the equation can be rearranged to give $x = \dfrac{7}{x^2 + 2}$ and explain why the iteration $x_{n+1} = \dfrac{7}{x_n^2 + 2}$ converges to $\alpha$. **[3]**

(ii) Starting from $x_0 = 1.5$, find $x_1, x_2, x_3, x_4$, giving your answers to 4 d.p. **[2]**

(iii) On a copy of the diagram below, draw a cobweb or staircase diagram to illustrate this iteration. **[3]**



**Fact —** There are four cases, determined by the gradient of $y = g(x)$ at the fixed point:

| **Gradient** $g'(\alpha)$ | **Pattern** | **Behaviour** |
|---|---|---|
| $0 < g'(\alpha) < 1$ | Staircase | Converges |
| $g'(\alpha) > 1$ | Staircase | Diverges |
| $-1 < g'(\alpha) < 0$ | Cobweb | Converges |
| $g'(\alpha) < -1$ | Cobweb | Diverges |

## Speed of Convergence

The error at step $n$ is $e_n = x_n - \alpha$, where $\alpha$ is the true root.

**Example**

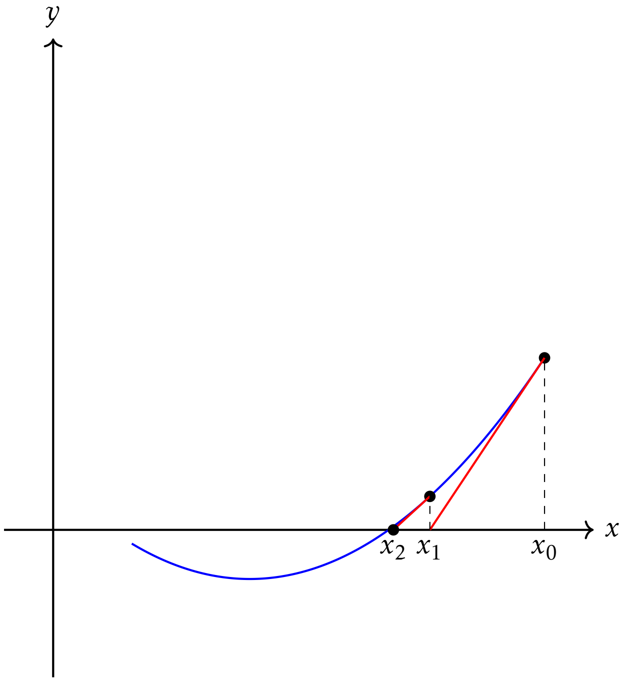
Show that for the iteration $x_{n+1} = g(x_n)$, the errors satisfy $e_{n+1} \approx g'(\alpha) \cdot e_n$.

**Example**

What happens if $g'(\alpha) = 0$? Show that the errors then satisfy $e_{n+1} \approx \frac{1}{2} g''(\alpha) \cdot e_n^2$.

## Newton–Raphson Method

We draw the tangent to $y = f(x)$ at our current estimate $x_n$ and take the $x$-intercept as $x_{n+1}$:




**Example**

Derive the Newton–Raphson formula: starting from the tangent at $(x_n, f(x_n))$, find $x_{n+1}$.

**Example**

Newton–Raphson is the iteration $x_{n+1} = g(x_n)$ where $g(x) = x - \dfrac{f(x)}{f'(x)}$. Show that $g'(\alpha) = 0$ at any simple root $\alpha$, and deduce that Newton–Raphson has quadratic convergence.

**Example**

Use the Newton–Raphson method to find the root of $f(x) = x^3 - 3x + 1$ near $x = 1.5$, correct to 4 decimal places.

**Example**

Use the Newton–Raphson method to find $\sqrt{5}$ to 4 decimal places.

**Example**

When can Newton–Raphson fail?

## Comparison of Methods

**Fact** (Summary) —

| Method | Pros | Cons |
|---|---|---|
| Bisection | Always works (if sign change exists); guaranteed to narrow interval | Slow (only one binary digit per step) |
| Decimal search | Easy to implement; systematic | Slow; lots of function evaluations |
| Linear interpolation | Faster than bisection | Can be slow if the curve is very flat on one side |
| Fixed-point iteration | Simple; can be fast | Rearrangement must satisfy $\lvert g'(\alpha)\rvert < 1$; may diverge |
| Newton–Raphson | Very fast convergence | Needs $f'(x)$; can fail at stationary points |

**Tip**

Whichever method you use, always finish by *verifying* the root to the required accuracy with a sign change!